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Fundamental Solution of a Three-dimensional Fractional Jeffreys-type Heat Equation

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Abstract. The three-dimensional Cauchy problem for the heat conduction equation with a fractional Jeffreys-type constitutive law is studied. Two different cases are distinguished: diffusion and propagation regimes. In the diffusion regime the three-dimensional fundamental solution is shown to be a spatial probability density function evolving in time. In the propagation regime the solution can have negative values, and therefore, does not allow a probabilistic interpretation. Explicit integral representation for the three-dimensional fundamental solution is derived and used for numerical experiments.

INTRODUCTION

This work is concerned with the multi-dimensional fractional Jeffreys-type heat conduction equation

$$\left(1 + \tau_q D_t^\alpha\right) \frac{\partial}{\partial t} T(\mathbf{x}, t) = \mathcal{D} \left(1 + \tau_T D_t^\alpha\right) \Delta T(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0, \tag{1}$$

where T is the temperature, τ_q and τ_T are relaxation times, $\mathcal{D}$ is the thermal diffusivity, $\mathbf{x}$ denotes the position vector, t is the temporal variable, D_t^α is the Riemann-Liouville fractional time-derivative of order α, $0 < \alpha \leq 1$, Δ denotes the spatial Laplace operator, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$.

The fractional Jeffreys-type equation (1) generalizes the classical Fourier heat conduction equation, obtained by setting $\tau_q = \tau_T = 0$, the Cattaneo (or telegrapher's) equation, obtained by setting $\tau_T = 0$ and $\alpha = 1$, and the classical Jeffreys-type heat equation, which corresponds to $\alpha = 1$, $\tau_q, \tau_T > 0$, $\tau_q \neq \tau_T$.

The fractional Jeffreys-type heat conduction equation is proposed in [1]. The one-dimensional Cauchy problem for equation (1) is solved analytically in ([2], Chapter 7) by means of the Laplace and Fourier transforms in the time and space domains, respectively. In [3] equation (1) is studied for $\tau_q > \tau_T$ in the context of unidirectional flows of viscoelastic fluids. In [4] an initial-value problem for the classical version ($\alpha = 1$) of the three-dimensional equation (1) is studied, where unphysical behavior of the solution with negative values is found for $\tau_q > \tau_T$. To address the shortcomings reported in [4] a fractional Jeffreys-type equation of a general form is proposed and studied in [5]. In [6] the fundamental solution to the Cauchy problem for equation (1) is studied analytically and numerically with the main emphasis on the one-dimensional case.

In this work we continue our study in [6] and analyze the behavior of the three-dimensional fundamental solution to the Cauchy problem for equation (1) with $n = 3$.

The paper is organized as follows. The next section contains preliminaries. In the following section the properties of the three-dimensional fundamental solution are studied based on a relation with the one-dimensional solution and on the properties of its Laplace transform. An explicit integral representation is derived and used for numerical experiments for different values of the parameters. The mean squared displacement is also derived. The last section contains concluding remarks.

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PRELIMINARIES

The Riemann–Liouville fractional derivative D_t^α of order $\alpha \in (0,1)$ is defined as [7]

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-t')^{-\alpha} u(t')\,\mathrm{d}t', \quad \alpha \in (0,1). \tag{2}$$

The Laplace transform of a function $u(t)$, $t \in \mathbb{R}_+$, is denoted by

$$\mathcal{L}\{u(t)\}(s) = \widehat{u}(s) = \int_0^\infty e^{-st} u(t)\,\mathrm{d}t.$$

For a function of several variables we denote by $\widehat{u}(\mathbf{x}, s)$ or $\mathcal{L}\{u\}(\mathbf{x}, s)$ the Laplace transform of $u(\mathbf{x}, t)$ with respect to t.

The Laplace transform pair related to the Riemann–Liouville fractional derivative is

$$\mathcal{L}\{D_t^\alpha u\}(s) = s^\alpha \mathcal{L}\{u\}(s), \quad \alpha \in (0,1), \tag{3}$$

provided u is continuous function satisfying $u(0) < \infty$ (see, e.g., [7], Chapter 1, Equation (1.29)).

The Fourier transform of a function $v(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is given by

$$\mathcal{F}\{v(\mathbf{x})\}(\mathbf{k}) = \widetilde{v}(\mathbf{k}) = \int_{\mathbb{R}^n} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} v(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \mathbf{k} \in \mathbb{R}^n.$$

The Fourier transform pair corresponding to the Laplace operator Δ of a function $v(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, such that $\lim_{|\mathbf{x}|\to\infty} v(\mathbf{x}) = 0$, is

$$\mathcal{F}\{\Delta v\}(\mathbf{k}) = -|\mathbf{k}|^2 \mathcal{F}\{v\}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^n. \tag{4}$$

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is said to be a completely monotone function ($\mathcal{CMF}$) if it is of class C^∞ and

$$(-1)^n \varphi^{(n)}(\lambda) \geq 0, \quad \lambda > 0, n = 0, 1, 2, ... \tag{5}$$

The characterization of the class $\mathcal{CMF}$ is given by the Bernstein's theorem, which states that a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure (non-negative function or generalized function).

A non-negative function $\varphi \in C^\infty(\mathbb{R}_+)$ is said to be a Bernstein function ($\mathcal{BF}$) if $\varphi'(\lambda) \in \mathcal{CMF}$.

For details on the classes $\mathcal{CMF}$ and $\mathcal{BF}$ and other related classes of functions we refer to [8, 9].

The two-parameter Mittag–Leffler function is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \ \Re\alpha > 0. \tag{6}$$

The Mittag–Leffler function satisfies the Laplace transform identity (see, e.g., [7], Equation (E.53))

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\omega t^\alpha)\right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \omega}. \tag{7}$$

The function $E_{\alpha,\beta}(-\lambda) \in \mathcal{CMF}$ for $\lambda > 0$ if $0 < \alpha \leq 1$ and $\beta \geq \alpha$ (see, e.g., [7], Equation (E.32)).

THREE-DIMENSIONAL FUNDAMENTAL SOLUTION AS A PROBABILITY DENSITY

We are concerned with the solution to the Cauchy problem for the fractional Jeffreys' heat conduction equation

$$\left(1 + \tau_q D_t^\alpha\right) \frac{\partial}{\partial t} T(\mathbf{x}, t) = \mathcal{D}\left(1 + \tau_T D_t^\alpha\right) \Delta T(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0, \tag{8}$$

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}); \quad \lim_{t\to 0^+} \frac{\partial}{\partial t} T(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^n, \tag{9}$$

$$\lim_{|\mathbf{x}|\to\infty} T(\mathbf{x}, t) = 0, \quad t > 0. \tag{10}$$

By applying Laplace transform with respect to the temporal variable and Fourier transform with respect to the spatial variables in equation (8) and taking into account initial conditions (9), the boundary condition (10), and identities (3) and (4) we derive the solution of the Cauchy problem in Fourier-Laplace domain

$$\widehat{\widetilde{T}}(\mathbf{k}, s) = \frac{g(s)/s}{g(s) + |\mathbf{k}|^2} \widetilde{T}_0(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^n, \quad s > 0, \tag{11}$$

where $g(s)$ denotes the following characteristic function

$$g(s) = \frac{s(1 + \tau_q s^\alpha)}{\mathcal{D}(1 + \tau_T s^\alpha)}, \quad s > 0. \tag{12}$$

Therefore, the solution of the Cauchy problem (8)-(9)-(10) is given by the integral

$$T(\mathbf{x}, t) = \int_{\mathbb{R}^3} \mathcal{G}_n(\mathbf{x} - \mathbf{y}, t) T_0(\mathbf{y}) \, \mathrm{d}^n\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \quad t > 0.$$

where $\mathcal{G}_n(\mathbf{x}, t)$ is the fundamental solution (Green function), which corresponds to $T_0(\mathbf{x}) = \delta(\mathbf{x}) = \prod_{j=1}^n \delta(x_j), \mathbf{x} \in \mathbb{R}^n$, with δ being the Dirac delta function. From (11) we deduce that the fundamental solution is defined in Fourier-Laplace domain as

$$\widehat{\widetilde{\mathcal{G}_n}}(\mathbf{k}, s) = \frac{g(s)/s}{g(s) + |\mathbf{k}|^2}, \quad \mathbf{k} \in \mathbb{R}^n, \quad s > 0. \tag{13}$$

Applying the formula for inverse Fourier transform, we deduce from (13)

$$\widehat{\mathcal{G}_n}(\mathbf{x}, s) = \frac{g(s)}{(2\pi)^n s} \int_{\mathbb{R}^n} \frac{e^{-\mathrm{i}\mathbf{k}.\mathbf{x}}}{g(s) + |\mathbf{k}|^2} \, \mathrm{d}^n\mathbf{k}, \quad \mathbf{x} \in \mathbb{R}^n, \quad s > 0. \tag{14}$$

Our goal now is to establish a relation between $\mathcal{G}_3(\mathbf{x}, t)$ and $\mathcal{G}_1(x, t)$ by the use of identity (14). In the case $n = 3$ we have in spherical coordinates $\mathrm{d}^3\mathbf{k} = k^2 \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta \, \mathrm{d}k$, where $k = |\mathbf{k}| \in (0, \infty)$, $\varphi \in (0, 2\pi)$, and $\theta \in (0, \pi)$ is taken to be the angle between $\mathbf{x}$ and $\mathbf{k}$. Thus, $\mathbf{k}.\mathbf{x} = kr\cos\theta$, where $r = |\mathbf{x}|$. Therefore

$$\widehat{\mathcal{G}_3}(\mathbf{x}, s) = \frac{g(s)}{(2\pi)^3 s} \int_{\mathbb{R}^3} \frac{e^{-\mathrm{i}\mathbf{k}.\mathbf{x}}}{g(s) + |\mathbf{k}|^2} \, \mathrm{d}^3\mathbf{k} = \frac{g(s)}{(2\pi)^2 s} \int_0^\infty \frac{k^2}{g(s) + k^2} \int_0^\pi \exp(-\mathrm{i}kr\cos\theta) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}k.$$

Taking into account that

$$\int_0^\pi \exp(-\mathrm{i}kr\cos\theta) \sin\theta \, \mathrm{d}\theta = \frac{e^{\mathrm{i}kr} - e^{-\mathrm{i}kr}}{\mathrm{i}kr}$$

it follows

$$\widehat{\mathcal{G}_3}(\mathbf{x}, s) = \frac{g(s)}{(2\pi)^2 rs} \int_{-\infty}^\infty \frac{\mathrm{i}ke^{-\mathrm{i}kr}}{g(s) + k^2} \, \mathrm{d}k, \quad r = |\mathbf{x}|. \tag{15}$$

On the other hand, in the one-dimensional case (14) implies

$$\widehat{\mathcal{G}_1}(x, s) = \frac{g(s)}{2\pi s} \int_{-\infty}^\infty \frac{e^{-\mathrm{i}kx}}{g(s) + k^2} \, \mathrm{d}k. \tag{16}$$

A comparison of (15) and (16) yields

$$\widehat{\mathcal{G}_3}(\mathbf{x}, s) = -\frac{1}{2\pi r} \frac{\partial}{\partial r} \widehat{\mathcal{G}_1}(r, s), \quad r = |\mathbf{x}|. \tag{17}$$

Applying the well-known formula

$$\mathcal{F}\{\exp(-a|x|)\}(k) = \frac{2a}{a^2 + k^2}, \quad a > 0; \quad x, k \in \mathbb{R},$$

we get from (11) the Laplace transform of the fundamental solution of the one-dimensional problem

$$\widehat{\mathcal{G}_1}(x, s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x| \sqrt{g(s)}\right), \quad x \in \mathbb{R}, s > 0. \tag{18}$$

Combining (17) and (18) yields

$$\widehat{\mathcal{G}_3}(\mathbf{x}, s) = \frac{g(s)}{4\pi|\mathbf{x}|s} \exp\left(-|\mathbf{x}| \sqrt{g(s)}\right), \quad \mathbf{x} \in \mathbb{R}^3, s > 0. \tag{19}$$

Applying inverse Laplace transform in (17) we obtain that an analogous relation holds true between the three-dimensional and the one-dimensional solutions:

$$\mathcal{G}_3(\mathbf{x}, t) = -\frac{1}{2\pi r} \frac{\partial}{\partial r} \mathcal{G}_1(r, t), \quad r = |\mathbf{x}|. \tag{20}$$

Let us deduce information about the behavior of the three-dimensional fundamental solution, based on relations (17), (18), (19), and (20). First, from (19) or (20) we see that $\mathcal{G}_3(\mathbf{x}, t)$ is a radial function, since its value at each point $\mathbf{x} \in \mathbb{R}^3$ depends only on the distance $r = |\mathbf{x}|$ between that point and the origin.

It is proven in [6] that the one-dimensional fundamental solution $\mathcal{G}_1(x, t)$ is a spatial probability density evolving in time, i.e. it is nonnegative and normalized:

$$\mathcal{G}_1(x, t) \geq 0, \quad \int_{\mathbb{R}} \mathcal{G}_1(x, t)\, \mathrm{d}x = 1, \quad x \in \mathbb{R},\ t \geq 0.$$

In this section we discuss the question whether the three-dimensional fundamental solution is a spatial probability density.

We first check that the three-dimensional solution $\mathcal{G}_3(\mathbf{x}, t)$ is always normalized, i.e. that it satisfies the identity

$$\int_{\mathbb{R}^3} \mathcal{G}_3(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x} = 1, \quad \mathbf{x} \in \mathbb{R}^3,\ t \geq 0, \tag{21}$$

which, by applying the uniqueness property of Laplace transform, is equivalent to

$$\int_{\mathbb{R}^3} \widehat{\mathcal{G}_3}(\mathbf{x}, s)\, \mathrm{d}^3\mathbf{x} = 1/s, \quad \mathbf{x} \in \mathbb{R}^3,\ s > 0. \tag{22}$$

Taking into account (19) and switching to spherical coordinates, we represent the last integral in the form

$$\begin{aligned}
\int_{\mathbb{R}^3} \widehat{\mathcal{G}_3}(\mathbf{x}, s)\, \mathrm{d}^3\mathbf{x} &= \frac{g(s)}{4\pi s} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}|} \exp\left(-|\mathbf{x}| \sqrt{g(s)}\right) \mathrm{d}^3\mathbf{x} \\
&= \frac{g(s)}{4\pi s} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{1}{r} \exp\left(-r \sqrt{g(s)}\right) r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi \\
&= \frac{g(s)}{s} \int_0^{\infty} r \exp\left(-r \sqrt{g(s)}\right) \mathrm{d}r = \frac{1}{s}, \quad s > 0.
\end{aligned}$$

For the last equality we have used the identity

$$\int_0^{\infty} r^{b-1} \exp(-ar)\, \mathrm{d}r = \frac{\Gamma(b)}{a^b}, \quad b > 0,\ a > 0, \tag{23}$$

which follows from the definition of the Gamma function $\Gamma(\cdot)$. In this way (22), and thus (21), is proved. Therefore, the three-dimensional fundamental solution is always normalized.

Further, in order to determine whether the three-dimensional solution is positive, we use relation (20) and the monotonicity properties of the one-dimensional solution. In [6] we presented plots of one-dimensional solutions for different values of the parameters and observed the following. The behavior of the one-dimensional solution is typical for a diffusion process when $\tau_q < \tau_T$: it is monotonically decreasing in x for $x > 0$. Therefore, according to (20), in this case the three-dimensional solution will be positive everywhere. On the other hand, for $\tau_q > \tau_T$ the behavior of the one-dimensional solution is typical for a propagation process, with a maximum moving away from the origin. That is, in the case $\tau_q > \tau_T$, the one-dimensional fundamental solution, considered as a function of x, $x > 0$, is initially an increasing function, attaining a maximum at $x* \neq 0$, and after that it decreases monotonically to 0 as $x \to \infty$. This, according to (20) implies negative values for the three-dimensional solution.

Next we prove strictly the above observations for the one-dimensional solution and the corresponding results for the three-dimensional solution.

First, let $\tau_q < \tau_T$. In this case $g(s) \in \mathcal{BF}$ and $\sqrt{g(s)} \in \mathcal{BF}$, see [6], Proposition 2. Therefore, $g(s)/s \in \mathcal{CMF}$ and $\exp(-a\sqrt{g(s)}) \in \mathcal{CMF}$ for $a > 0$. Since the class $\mathcal{CMF}$ is closed under point-wise multiplication, it follows

$$\frac{g(s)}{s}\exp(-a\sqrt{g(s)}) \in \mathcal{CMF}, \quad a > 0, \quad \tau_q < \tau_T. \tag{24}$$

Therefore (19) implies $\widehat{\mathcal{G}_3}(\mathbf{x}, s) \in \mathcal{CMF}$ for $s > 0$ when $|\mathbf{x}|$ is considered as a parameter. This, according to Bernstein's theorem implies the nonnegativity of $\mathcal{G}_3(\mathbf{x}, t)$ for $\tau_q < \tau_T$. Therefore, in the diffusion regime $\tau_q < \tau_T$ the three-dimensional fundamental solution is a spatial probability density function evolving in time.

Due to relation (20) the three-dimensional solution $\mathcal{G}_3(\mathbf{x}, t)$ is nonnegative if and only if the one-dimensional solution is monotonically decreasing in x for $x > 0$. Next we show that in the case $\tau_q > \tau_T$ the solution $\mathcal{G}_1(x, t)$ is an increasing function of x near the origin, for all $t > 0$. Indeed, differentiation of (18) yields

$$\mathcal{L}\left\{\frac{\partial \mathcal{G}_1}{\partial x}\right\}(x, s) = \frac{\partial}{\partial x}\widehat{\mathcal{G}_1}(x, s) = -\frac{g(s)}{2s}\exp\left(-x\sqrt{g(s)}\right), \quad x \geq 0, s > 0.$$

Therefore

$$\lim_{x\to 0+}\frac{\partial}{\partial x}\widehat{\mathcal{G}_1}(x, s) = -\frac{g(s)}{2s} = -\frac{1 + \tau_q s^\alpha}{2\mathcal{D}(1 + \tau_T s^\alpha)} = -\frac{\tau_q}{2\mathcal{D}\tau_T}\left(\frac{\tau_T - \tau_q}{\tau_q \tau_T}\frac{1}{s^\alpha + 1/\tau_T} + 1\right),$$

which, after inverting the Laplace transform by the use of (7) and $\widehat{\delta} = 1$ yields

$$\lim_{x\to 0+}\frac{\partial}{\partial x}\mathcal{G}_1(x, t) = -\frac{\tau_q}{2\mathcal{D}\tau_T}\left(\frac{\tau_T - \tau_q}{\tau_q \tau_T}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\tau_T}\right) + \delta(t)\right)$$

Therefore, due to the fact that the Mittag-Leffler function is positive, it follows for all $t > 0$

$$\lim_{x\to 0+}\frac{\partial}{\partial x}\mathcal{G}_1(x, t) > 0, \quad \tau_q > \tau_T.$$

Taking into account relation (20) this implies that in the case $\tau_q > \tau_T$ the three-dimensional solution is negative near the origin, $|\mathbf{x}| < r*$.

The behavior of the fundamental solution is illustrated in the next section.

INTEGRAL REPRESENTATION OF THE FUNDAMENTAL SOLUTION AND PLOTS

Explicit integral representation for $\mathcal{G}_3(\mathbf{x}, t)$ can be derived by applying the complex Laplace inversion formula to (19), which gives:

$$\mathcal{G}_3(\mathbf{x}, t) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{st}\widehat{\mathcal{G}_3}(\mathbf{x}, s)\,\mathrm{d}s = \frac{1}{8\pi^2 \mathrm{i}|\mathbf{x}|}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{g(s)}{s}\exp\left(st - |\mathbf{x}|\sqrt{g(s)}\right)\mathrm{d}s, \quad \gamma > 0. \tag{25}$$

The function under the integral sign in (25) can be analytically extended to $\mathbb{C}\backslash(-\infty, 0]$. For the multivalued function $s^\alpha = \exp(\alpha \ln s)$ the principal branch is considered. By the Cauchy's theorem, the integration on the Bromwich path $(\gamma - \mathrm{i}\infty, \gamma + \mathrm{i}\infty)$ can be replaced by integration on the contour $D \cup D_0$, where

$$D = \{s = \mathrm{i}r,\ r \in (-\infty, -\varepsilon) \cup (\varepsilon, \infty)\},\ D_0 = \{s = \varepsilon e^{\mathrm{i}\theta}, \theta \in [-\pi/2, \pi/2]\}.$$

Indeed, the integrals on the contours $\{s = \sigma \pm \mathrm{i}R, \sigma \in (0, \gamma)\}$ vanish for $R \to \infty$ due to the following asymptotic expression

$$|g(s)| \approx \frac{\tau_q}{\tau_T}|s| = \frac{\tau_q}{\tau_T}(\sigma^2 + R^2)^{1/2}, \quad R \to \infty,$$

and

$$\Re\sqrt{g(s)} \approx \sqrt{\frac{\tau_q}{\tau_T}|s|}\cos\frac{\arg s}{2} \approx \left(\frac{\tau_q}{\tau_T}(\sigma^2 + R^2)^{1/2}\right)^{1/2}\cos(\pm\pi/4), \quad R \to \infty.$$

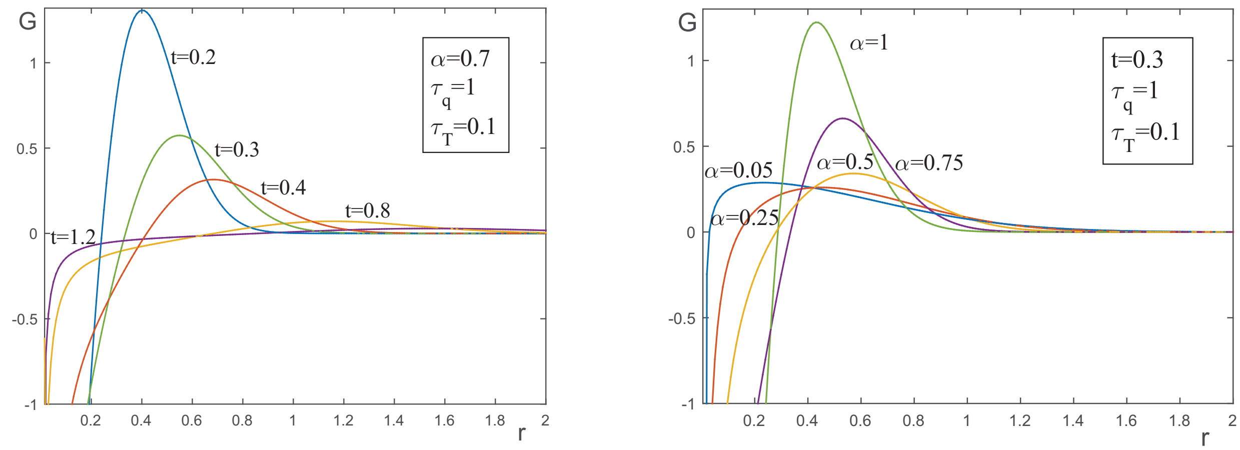

FIGURE 1. The fundamental solution $\mathcal{G}_3(\mathbf{x}, t)$ versus $r = |\mathbf{x}|$ for $\alpha = 0.7$, $\tau_q = 1$, $\tau_T = 0.1$, and $t = 0.2, 0.3, 0.4, 0.8, 1.2$ (left); for $t = 0.3$, $\tau_q = 1$, $\tau_T = 0.1$, and $\alpha = 0.05, 0.25, 0.5, 0.75, 1$ (right).

The integral on the semi-circular contour D_0 also vanishes for $\varepsilon \to 0$ due to the limit

$$\lim_{s\to 0} s\left(\frac{g(s)}{s}\exp\left(st - |x|\sqrt{g(s)}\right)\right) = \lim_{s\to 0} g(s) = 0.$$

Therefore, integration on the contour D yields after letting $\varepsilon \to 0$

$$\mathcal{G}_3(\mathbf{x}, t) = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} e^{st}\widehat{\mathcal{G}_3}(\mathbf{x}, s)\,\mathrm{d}s = \frac{1}{4\pi^2|\mathbf{x}|}\int_0^\infty \Im\left(e^{\mathrm{i}\sigma t} g(\mathrm{i}\sigma)\exp\left(-|\mathbf{x}|\sqrt{g(\mathrm{i}\sigma)}\right)\right)\frac{\mathrm{d}\sigma}{\sigma}.$$

To get an explicit expression it remains to find the imaginary part of the function under the integral sign. We apply the formula for the real and imaginary parts of the square root of a complex number and obtain after some standard manipulations the following result for $\mathbf{x} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$ and $t > 0$:

$$\mathcal{G}_3(\mathbf{x}, t) = \frac{1}{4\pi^2|\mathbf{x}|}\int_0^\infty \exp\left(-|\mathbf{x}|K^-(\sigma)\right)\left(-A(\sigma)\sin\left(\sigma t - |\mathbf{x}|K^+(\sigma)\right) + B(\sigma)\cos\left(\sigma t - |\mathbf{x}|K^+(\sigma)\right)\right)\mathrm{d}\sigma, \tag{26}$$

where $K^{\pm}(\sigma)$ are defined by

$$K^{\pm}(\sigma) = \left(\frac{\sigma}{2}\right)^{1/2}\left(\left(A^2(\sigma) + B^2(\sigma)\right)^{1/2} \pm A(\sigma)\right)^{1/2} \tag{27}$$

with

$$A(\sigma) = \frac{(\tau_q - \tau_T)\sigma^\alpha \sin(\alpha\pi/2)}{\mathcal{D}\left(1 + 2\tau_T\sigma^\alpha\cos(\alpha\pi/2) + \tau_T^2\sigma^{2\alpha}\right)}, \quad B(\sigma) = \frac{1 + (\tau_q + \tau_T)\sigma^\alpha\cos(\alpha\pi/2) + \tau_q\tau_T\sigma^{2\alpha}}{\mathcal{D}\left(1 + 2\tau_T\sigma^\alpha\cos(\alpha\pi/2) + \tau_T^2\sigma^{2\alpha}\right)}. \tag{28}$$

An alternative way to deduce (26) is to use relation (20) and the formula for the one-dimensional fundamental solution $\mathcal{G}_1(x, t)$, derived in [6], Theorem 2.

The integral representation (26) is used for numerical computation and visualization of the fundamental solution $\mathcal{G}_3(\mathbf{x}, t)$ for different values of the parameters. For the numerical computation of the improper integral in (26) the MATLAB subroutine "integral" is used.

Some plots of the fundamental solution $\mathcal{G}_3(\mathbf{x}, t)$ with $\mathcal{D} = 1$ are presented in Figures 1 and 2. The presented plots concern only the propagation regime $\tau_q > \tau_T$, more precisely $\tau_q = 1, \tau_T = 0.1$.

In Figure 1 the fundamental solution $\mathcal{G}_3(\mathbf{x}, t)$ is plotted as a function of $r = |\mathbf{x}|$ for different values of the temporal variable t (left) and for different values of the fractional order α (right), including the classical case $\alpha = 1$. In Figure 2 the solution is plotted as a function of t for different values of $r = |\mathbf{x}|$.

All of the presented plots confirm that in the case $\tau_q > \tau_T$ the solution $\mathcal{G}_3(\mathbf{x}, t)$ admits negative values for sufficiently small $r = |\mathbf{x}|$.

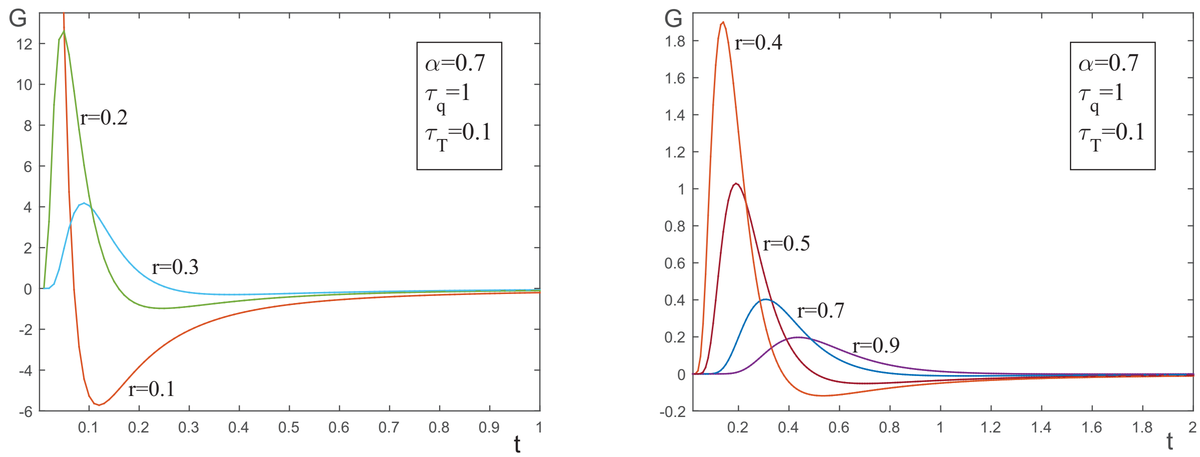

FIGURE 2. The fundamental solution $\mathcal{G}_3(\mathbf{x}, t)$ versus t for $\alpha = 0.7$, $\tau_q = 1$, $\tau_T = 0.1$, and different values of $r = |\mathbf{x}|$: $r = 0.1, 0.2, 0.3$ (left); $r = 0.4, 0.5, 0.7, 0.9$ (right).

MEAN SQUARED DISPLACEMENT

Let us first calculate the moments of the solution $\mathcal{G}_3(\mathbf{x}, t)$ of order β, i.e. the integrals

$$I_\beta(t) = \int_0^\infty r^\beta \mathcal{G}_3(r, t)\, \mathrm{d}r, \quad \beta > 0.$$

In Laplace domain we obtain by applying (19) and (23)

$$\widehat{I_\beta}(s) = \int_0^\infty r^\beta \widehat{\mathcal{G}_3}(r, t)\, \mathrm{d}r = \frac{g(s)}{4\pi s} \int_0^\infty r^{\beta-1} \exp\left(-r\sqrt{g(s)}\right) \mathrm{d}r = \frac{\Gamma(\beta) g(s)^{1-\beta/2}}{4\pi s}, \quad \beta > 0, \tag{29}$$

where $g(s)$ is the characteristic function (12).

Let us calculate the mean squared displacement (MSD). In Laplace domain by switching to spherical coordinates and using (29) we deduce

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \widehat{\mathcal{G}_3}(\mathbf{x}, s)\, \mathrm{d}^3\mathbf{x} = \frac{g(s)}{4\pi s} \int_{\mathbb{R}^3} |\mathbf{x}| \exp\left(-|\mathbf{x}|\sqrt{g(s)}\right) \mathrm{d}^3\mathbf{x} = \frac{g(s)}{4\pi s} \int_0^{2\pi}\int_0^{\pi}\int_0^\infty r^3 \exp\left(-r\sqrt{g(s)}\right) \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi = 4\pi \widehat{I_4}(s).$$

Therefore,

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \widehat{\mathcal{G}_3}(\mathbf{x}, s)\, \mathrm{d}^3\mathbf{x} = \frac{6}{s g(s)}. \tag{30}$$

On the other hand, in the case of one spatial dimension we have by the use of (18) and (23)

$$\int_{\mathbb{R}} x^2 \widehat{\mathcal{G}_1}(x, s)\, dx = \frac{\sqrt{g(s)}}{s} \int_0^\infty x^2 \exp\left(-x\sqrt{g(s)}\right) \mathrm{d}x = \frac{2}{s g(s)}.$$

Therefore

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \mathcal{G}_3(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x} = 3 \int_{\mathbb{R}} |x|^2 \mathcal{G}_1(x, t)\, \mathrm{d}x.$$

This relation implies that the three-dimensional MSD differs from its one-dimensional counterpart by a factor of 3 and therefore has the same qualitative behavior as the one-dimensional MSD. By inversion of the Laplace transform in (30) we deduce by the use of identity (7) (see also Eq. (44) in [6])

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \mathcal{G}_3(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x} = 6\frac{\tau_T}{\tau_q} t + \frac{6}{\tau_q}\left(1 - \frac{\tau_T}{\tau_q}\right) t^{\alpha+1} E_{\alpha,\alpha+2}\left(-\frac{t^\alpha}{\tau_q}\right).$$

CONCLUDING REMARKS

In this paper we study the fundamental solution for the three-dimensional fractional Jeffreys-type heat equation. While in the diffusion regime ($\tau_q < \tau_T$) it is a spatial probability density function evolving with time, in the propagation regime ($\tau_q > \tau_T$) it is negative for sufficiently small $|\mathbf{x}|$ and, therefore, does not allow a probabilistic interpretation.

A strong analogy is observed between the established properties of the three-dimensional fundamental solution to the Jeffreys-type heat equation and those of the fractional diffusion-wave equation with the Caputo time-derivative (see e.g. [10]) with its two different regimes: diffusion and propagation (corresponding to time-derivative of order $0 < \beta < 1$ and $1 < \beta < 2$, respectively).

The technique developed in this work can be applied to generalized time-fractional diffusion-wave equations (for a definition see e.g. [11]) with a general characteristic function $g(s)$, satisfying $g(s) \in \mathcal{BF}$ in the diffusion regime and $\sqrt{g(s)} \in \mathcal{BF}$, $g(s) \notin \mathcal{BF}$ in the propagation regime.

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REFERENCES

[1] T. M. Atanacković, A. Grillo, G. Wittum, and D. Zorica, “Fractional Jeffreys-type diffusion equation,” in *Proceedings of 4th IFAC Workshop on Fractional Differentiation and Its Applications* (University of Extremadura, Badajoz, Spain, 18-20 October, 2010).

[2] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes* (John Wiley & Sons, London, 2014).

[3] E. Bazhlekova and I. Bazhlekov, Comput. Math. Appl. **73**, 1363–1376 (2017).

[4] S. Rukolaine, Int. J. Heat Mass Tran. **78**, 58–63 (2014).

[5] E. Awad, Int. J. Heat Mass Tran. **158**, p. 119742 (2020).

[6] E. Bazhlekova and I. Bazhlekov, Fractal Fract. **4**, p. 32 (2020).

[7] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity* (Imperial College Press, London, 2010).

[8] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions: Theory and Applications* (De Gruyter, Berlin, 2010).

[9] J. Prüss, *Evolutionary Integral Equations and Applications* (Birkhäuser, Basel, 1993).

[10] A. Hanyga, Proc. R. Soc. Lond. A **458**, 933–957 (2002).

[11] E. Bazhlekova, Fract. Calc. Appl. Anal. **21**, 869–900 (2018).